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Affine Schubert classes, Schur positivity, and combinatorial Hopf algebras

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Abstract

We develop the point of view that Schubert classes of the affine Grassmannian of a simple algebraic group G can be identified with Schur-positive symmetric functions. In particular, we give a geometric proof of the Schur positivity of k-Schur functions at $t = 1$, together with branching positivity results for the type C k-Schur functions. Our work is placed in the context of combinatorial Hopf algebras.

1. *Affine Schubert classes 'are' Schur-positive symmetric functions*

Let G be a simple and simply connected complex algebraic group, and let Gr_G denote the (ind-scheme) affine Grassmannian of G. Let W denote the Weyl group of G and W_{af} denote the affine Weyl group of G. The homology $H_*(\mathrm{Gr}_G) = H_*(\mathrm{Gr}_G, \mathbb{Z})$ has a basis given by the Schubert classes $\{\xi_w \mid w \in W_{\mathrm{af}}/W\}$ [**6**]. All (co)homologies have $\mathbb{Z}$-coefficients.

If $\iota : G \to G'$ is an inclusion of algebraic groups, then there is a closed embedding $\iota_{\mathrm{Gr}} : \mathrm{Gr}_G \to \mathrm{Gr}_{G'}$, see, for example [**5**, A.5]. The results of Kumar and Nori [**7**, Proposition (5)] imply that the homology class $[X] \in H_*(\mathrm{Gr}_{G'})$ of any finite-dimensional subvariety $X \subset \mathrm{Gr}_{G'}$ is a (finite) non-negative sum $[X] = \sum_w a_w \xi_w$ of the Schubert classes $\xi_w \in H_*(\mathrm{Gr}_{G'})$. Applying this to the image $\iota_{\mathrm{Gr}}(X_v) \subset \mathrm{Gr}_{G'}$ of a Schubert variety $X_v \subset \mathrm{Gr}_G$, we obtain the following theorem.

Theorem 1. *For any $v \in W_{\mathrm{af}}/W$, the pushforward $(\iota_{\mathrm{Gr}})_*(\xi_v) \in H_*(\mathrm{Gr}_{G'})$ of a Schubert class of Gr_G is a non-negative linear combination of Schubert classes $\{\xi_w \mid w \in W'_{\mathrm{af}}/W'\}$ of $\mathrm{Gr}_{G'}$.*

This simple observation was obtained as a consequence of discussions with Mark Shimozono, and is the basis for the current article. In the 'limit' $G = \mathrm{SL}(\infty, \mathbb{C})$ the homology $H_*(\mathrm{Gr}_{\mathrm{SL}(\infty,\mathbb{C})})$ can be identified with the ring Sym of symmetric functions, and the Schubert basis is given by the Schur functions. Indeed, $\mathrm{Gr}_{\mathrm{SL}(\infty,\mathbb{C})}$ is homotopy equivalent to $\Omega\mathrm{SU}(\infty)$ [**14**] and by Bott-periodicity [**3**] also to the classifying space $BU(\infty)$. It is well known that $H_*(BU(\infty)) \simeq \mathrm{Sym}$. Picking an embedding $G \hookrightarrow \mathrm{SL}(m, \mathbb{C}) \hookrightarrow \mathrm{SL}(\infty, \mathbb{C})$, one obtains a map $\eta : H_*(\mathrm{Gr}_G) \to \mathrm{Sym}$, and from Theorem 1 we see that 'every affine Schubert class is a Schur-positive symmetric function'. This statement is slightly misleading (hence the quotation marks) because for groups G with torsion, one cannot always arrange for η to be an inclusion (see Remark 1). Nevertheless, this setup is a potent explanation for many kinds of Schur-positivity.

In the cases $G = \mathrm{SL}(n, \mathbb{C})$ the homology ring $H_*(\mathrm{Gr}_G)$ is isomorphic to a subalgebra $\Lambda_{(n)}$ of symmetric functions, and it was shown in [**8**] that under this isomorphism the Schubert

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basis is identified with the k-Schur functions $s_\lambda^{(k)}(X)$ [11, 12] (with $k = n-1$ and $t = 1$) of Lapointe, Lascoux, and Morse. It was conjectured in [11] that the k-Schur functions expand positively in terms of $(k+1)$-Schur functions (called *k-branching positivity*), and in particular that they expand positively in terms of Schur functions (corresponding to $k = \infty$). These conjectures were motivated by the Macdonald positivity conjecture. In Section 2, we check that the map on homology induced by the natural inclusions $\iota : \mathrm{SL}(n,\mathbb{C}) \to \mathrm{SL}(n+1,\mathbb{C})$ is the natural inclusion in symmetric functions, obtaining these conjectures as consequences of Theorem 1. Recently, Assaf and Billey have announced a combinatorial proof of the Schur positivity conjecture, in the more general case, when the parameter t (not discussed here) is present. Another combinatorial approach to the k-branching positivity will be given in joint work [9] with Lapointe, Morse, and Shimozono.

In [10], the homology $H_*(\mathrm{Gr}_{\mathrm{Sp}(2n,\mathbb{C})})$ was also identified with a ring of symmetric functions, and the Schubert basis constructed. The inclusions $\mathrm{Sp}(2n,\mathbb{C}) \hookrightarrow \mathrm{Sp}(2n+2,\mathbb{C})$ and $\mathrm{Sp}(2n,\mathbb{C}) \hookrightarrow \mathrm{SL}(2n,\mathbb{C})$ give rise to other branching positivity and Schur-positivity statements.

Our point of view so far puts symmetric functions as the 'universal target' for the homologies of affine Grassmannians. It is natural to ask how much flexibility there is with the maps $\eta : H_*(\mathrm{Gr}_G) \to \mathrm{Sym}$ from our homology rings to symmetric functions. In Section 3, we connect this question with the category of combinatorial Hopf algebras studied by Aguiar, Bergeron, and Sottile [1]. It is shown in [1] that the Hopf algebra of symmetric functions is the terminal object in the category of cocommutative graded Hopf algebras equipped with a character. All the homologies $H_*(\mathrm{Gr}_G)$ are cocommutative graded Hopf algebras, so our aim to write affine Schubert classes as symmetric functions is very natural from this perspective. In Section 3, we give a topological interpretation of the theorem of [1], replacing Hopf algebras with H-spaces, and the character with a $U(\infty)$-bundle.

2. *Based loop spaces and symmetric functions*

Let $K \subset G$ denote the maximal compact subgroup. Let $LG = G(\mathbb{C}[z,z^{-1}])$ denote the space of algebraic maps $\mathbb{C}^* \to G$, and let $L^+G = G(\mathbb{C}[z])$ denote the space of algebraic maps $\mathbb{C} \to G$. Let $\Omega K \subset LG$ be the space of polynomial-based loops into K: these are maps $f : \mathbb{C}^* \to G$ the restriction to $S^1 \subset \mathbb{C}^*$ of which lie in K, and such that $f(1) = 1$. The affine Grassmannian Gr_G can be presented as LG/L^+G, and it is shown in [14, Section 8] that ΩK can be identified with Gr_G as topological spaces. In particular, we have $H_*(\Omega K) \simeq H_*(\mathrm{Gr}_G)$. Note that our affine Grassmannian, which is an ind-scheme, is denoted $\mathrm{Gr}_0^{\mathfrak{g}}$ in [14].

Now let $\iota : G \to G'$ be an inclusion. We will assume that the maximal compact subgroups are chosen so that $\iota(K) \subset K'$. To calculate the map $(\iota_{\mathrm{Gr}})_* : H_*(\mathrm{Gr}_G) \to H_*(\mathrm{Gr}_{G'})$, we will instead consider the map $H_*(\Omega K) \to H_*(\Omega K')$. We may do so because the identification $\Omega K \simeq LG/L^+G = \mathrm{Gr}_G$ is induced by the inclusion $\Omega K \subset LG$, and we have the commutative diagram:

$$\begin{tikzcd} \Omega K \arrow[r] \arrow[d] & \Omega K' \arrow[d] \\ LG \arrow[r] & LG'. \end{tikzcd}$$

Since $\Omega K \to \Omega K'$ is a map of groups, the map $(\iota_{\mathrm{Gr}})_* : H_*(\mathrm{Gr}_G) \to H_*(\mathrm{Gr}_{G'})$ is a Hopf-morphism.

The maximal compact subgroup of $\mathrm{SL}(n,\mathbb{C})$ is the special unitary group $\mathrm{SU}(n)$. We shall always consider $\mathrm{Sp}(2n,\mathbb{C}) \subset \mathrm{SL}(2n,\mathbb{C})$ in the natural way, and the maximal compact subgroup is denoted $\mathrm{Sp}(n) = \mathrm{SU}(2n) \cap \mathrm{Sp}(2n,\mathbb{C})$. We shall consider the following three inclusions: (a) $\mathrm{SL}(n,\mathbb{C}) \hookrightarrow \mathrm{SL}(n+1,\mathbb{C})$ giving $\mathrm{SU}(n) \hookrightarrow \mathrm{SU}(n+1)$, (b) $\mathrm{Sp}(2n,\mathbb{C}) \hookrightarrow \mathrm{Sp}(2n+2,\mathbb{C})$ giving $\mathrm{Sp}(n) \hookrightarrow \mathrm{Sp}(n+1)$, and (c) $\mathrm{Sp}(2n,\mathbb{C}) \hookrightarrow \mathrm{SL}(2n,\mathbb{C})$ giving $\mathrm{Sp}(n) \hookrightarrow \mathrm{SU}(2n)$. The inclusions (a) and (b) are the natural ones, induced by inclusions of coordinate subspaces. We shall assume that these embeddings are compatible, that is, the two compositions $\mathrm{Sp}(2n,\mathbb{C}) \hookrightarrow \mathrm{Sp}(2n+2,\mathbb{C}) \hookrightarrow \mathrm{SL}(2n+2,\mathbb{C})$ and $\mathrm{Sp}(2n,\mathbb{C}) \hookrightarrow \mathrm{SL}(2n,\mathbb{C}) \hookrightarrow \mathrm{SL}(2n+2,\mathbb{C})$ are identical. This is easy to achieve (see [**13**, p. 183]).

We now switch to the notation $H_*(\Omega K)$, instead of $H_*(\mathrm{Gr}_G)$. The homology ring $H_*(\Omega\mathrm{SU}(n))$ and $H_*(\Omega\mathrm{Sp}(n))$ are polynomial algebras with generators in dimensions $2, 4, \ldots, 2n-2$ and $2, 6, \ldots, 4n-2$, respectively.

Proposition 1. *The induced maps on homology*

$$\begin{aligned} H_*(\Omega\mathrm{SU}(n)) &\longrightarrow H_*(\Omega\mathrm{SU}(n+1)),\\ H_*(\Omega\mathrm{Sp}(n)) &\longrightarrow H_*(\Omega\mathrm{Sp}(n+1)),\\ H_*(\Omega\mathrm{Sp}(n)) &\longrightarrow H_*(\Omega\mathrm{SU}(2n)) \end{aligned}$$

are Hopf-inclusions. The first two maps are $\mathbb{Z}$-module isomorphisms below degrees $2n-1$ and $4n-1$, respectively.

Proof. That $H_*(\Omega\mathrm{SU}(n)) \to H_*(\Omega\mathrm{SU}(n+1))$ is injective is shown in [**3**, Proposition 8.4], by considering the fibering $\Omega\mathrm{SU}(n) \to \Omega\mathrm{SU}(n+1) \to \Omega S^{2n-1}$ (the homology is concentrated in even degrees in both base and fiber). The same argument, with ΩS^{4n-1} replacing ΩS^{2n-1} shows that $H_*(\Omega\mathrm{Sp}(n)) \to H_*(\Omega\mathrm{Sp}(n+1))$ is injective. The claims concerning isomorphisms in low dimensions also follow from this calculation.

Since all the inclusions are compatible, to show that $H_*(\Omega\mathrm{Sp}(n)) \to H_*(\Omega\mathrm{SU}(2n))$ is injective it suffices to show that $H_*(\Omega\mathrm{Sp}(\infty)) \to H_*(\Omega\mathrm{SU}(\infty)$ is injective. The corresponding cohomology map $H^*(\Omega\mathrm{SU}(\infty)) \simeq H^*(BU(\infty)) \to H^*(\mathrm{Sp}(\infty)/U(\infty)) \simeq H^*(\Omega\mathrm{Sp}(\infty))$ is calculated in [**13**, 3.18], and is manifestly surjective (see also [**13**, Lemma 5.2 and Theorem 5.14]). Thus, the dual map in homology is injective. □

Remark 1. For groups G with torsion, it may not be possible to find a Hopf-inclusion $H_*(\Omega K) \to H_*(\Omega\mathrm{SU}(m))$. Let $K = \mathrm{SO}(n)$ be the special orthogonal group. $\mathrm{SO}(n)$ is not simply connected, but we have $\Omega_0\mathrm{SO}(n) \simeq \Omega\mathrm{Spin}(n)$ where $\Omega_0\mathrm{SO}(n)$ denotes the connected component of the identity. According to [**3**, Proposition 10.1], the Hopf algebra $H^*(\Omega_0\mathrm{SO}(4n))$ has a primitive subspace of dimension 2 in degree $4n-2$, so admits no Hopf-inclusion into $H_*(\Omega\mathrm{SU}(m))$ that has primitive subspaces of at most dimension 1 in each degree.

We now calculate the inclusions of Proposition 1 in terms of symmetric functions. Presumably the following calculations are well known to topologists; we present them in a form emphasizing the connections with symmetric functions.

For symmetric function definitions we use here, we refer the reader to [**8**, **10**]. Let Sym denote the ring of symmetric functions in infinitely many variables $x_1, x_2, \ldots$, over $\mathbb{Z}$. We let h_i denote the homogeneous symmetric functions, e_i denote the elementary symmetric functions, and p_i denote the power sum symmetric functions. We let s_λ denote a Schur function. We let $\omega : \mathrm{Sym} \to \mathrm{Sym}$ denote the conjugation involution of Sym, sending h_i to e_i. The comultiplication of Sym is given by $\Delta(h_i) = \sum_{j=0}^{i} h_j \otimes h_{i-j}$, where $h_0 := 1$, or by $\Delta(p_i) = 1 \otimes p_i + p_i \otimes 1$.

2.1. *k-branching in* $\mathrm{SL}(n,\mathbb{C})$

The Hopf-subalgebra $\mathbb{Z}[h_1, h_2, \ldots, h_{n-1}]$ is Hopf-isomorphic to $H_*(\Omega\mathrm{SU}(n))$ and under this isomorphism we showed in [**8**] that the Schubert basis $\{\xi_w \in H_*(\Omega\mathrm{SU}(n)) \mid w \in W_{\mathrm{af}}/W\}$ is identified with the k-Schur functions $s_\lambda^{(k)} \in \mathrm{Sym}$ of [**12**], with $k = n-1$. For k larger than the degree, a k-Schur function is simply a Schur function. There is some flexibility in this isomorphism: one may compose with the involution ω, which sends k-Schur functions to k-Schur functions. At the level of the Weyl group, this corresponds to the non-trivial Dynkin diagram automorphism of $\tilde{A}_{n-1}$ which fixes the affine node 0. Note that the degree of a homogeneous symmetric function is half the topological degree of the corresponding homology class.

Let

$$\phi : \mathbb{Z}[h_1, h_2, \dots, h_{n-1}] \simeq H_*(\Omega \mathrm{SU}(n)) \longrightarrow H_*(\Omega \mathrm{SU}(n+1)) \simeq \mathbb{Z}[h_1, h_2, \dots, h_n]$$

be the Hopf-inclusion induced by Proposition 1. Since $H_2(\Omega \mathrm{SU}(n+1))$ has rank 1, we must have by Theorem 1 and Proposition 1 $\phi(h_1) = h_1$. Since ϕ is a Hopf-morphism, it must send primitive elements to primitive elements. The primitive elements are exactly the power sum symmetric functions $p_1, p_2, \dots$. Since ϕ is an isomorphism in low dimensions, it must send each power sum symmetric function p_i $(1 \leqslant i \leqslant n-1)$ to $\pm p_i$. We have two choices: $\phi(p_2) = \pm p_2$. We may assume, by possibly composing the identification $H_*(\Omega \mathrm{SU}(n+1)) \simeq \mathbb{Z}[h_1, h_2, \dots, h_n]$ with ω, that $\phi(p_2) = p_2$. Now suppose that we have established $\phi(p_i) = p_i$ for all $i < j$, for some $j > 2$. Expressing e_j as a polynomial in power sum symmetric functions, p_j occurs with coefficient $(-1)^j p_j/j$. We see that $e_j \pm 2p_j/j$ has monomials with fractional coefficients, so it does not lie in $\mathbb{Z}[h_1, h_2, \dots, h_n]$, and conclude that $\phi(p_j) = p_j$. It follows by induction that ϕ is the obvious inclusion.

From Theorem 1, we obtain the following corollary.

Corollary 1. *Every k-Schur function is $(k+1)$-Schur positive. In particular, k-Schur functions are Schur positive.*

2.2. *k-branching in* $\mathrm{Sp}(2n, \mathbb{C})$

Let $P_i \in \mathrm{Sym}$ denote the Schur P-functions labeled by a single row. In [**10**], we showed that one has a Hopf-isomorphism $H_*(\Omega \mathrm{Sp}(n)) \simeq \mathbb{Z}[P_1, P_3, \dots, P_{2n-1}]$, identifying the homology Schubert basis $\{\xi_w \in H_*(\Omega \mathrm{Sp}(n)) \mid w \in W_{\mathrm{af}}/W\}$ with symmetric functions denoted $P_w^{(n)}$ (type C k-Schur functions). The following result is implicit, but not completely spelt out in [**10**], so we do so here.

Proposition 2. *Let $w \in W_{\mathrm{af}}$ be a minimal coset representative in W_{af}/W. For $n > \ell(w)$, the symmetric function $P_w^{(n)}$ is a Schur P-function.*

Proof. We preserve all the notation of [**10**]. Using duality, it suffices to show that the symmetric functions $Q_w^{(n)}$ of [**10**] coincide with the Schur Q-functions when n is large. In this case, a reduced expression of w does not involve the simple generator s_n. (The simple generators of W_{af} are $s_0, s_1, \dots, s_n$, where s_0 is the affine node.) Thus in the formula [**10**, (1.2)] for $Q_w^{(n)}$, all $v \in W_{\mathrm{af}}$ involving s_n may be ignored. Comparing the definition of Z-s in [**10**] with [**4**, (4.1)], we see that our $Q_w^{(n)}$ are a power of 2 times the B_n-Stanley symmetric functions of [**4**]. The latter are known to be Schur P-functions [**4**, Theorem 8.2] in the special case of a Grassmannian B_n-element. It follows that the $Q_w^{(n)}$ of [**10**] are Schur Q-functions. $\square$

Note that one has $\mathbb{Q}[P_1, P_3, \dots, P_{2n-1}] \simeq \mathbb{Q}[p_1, p_3, \dots, p_{2n-1}]$. We have the formula

$$P_i = \frac{1}{2} \sum_{j=0}^{i} e_j h_{i-j}, \tag{2.1}$$

which gives a symmetric function with integral coefficients, despite the half. When written as a polynomial in power sum symmetric functions, only the terms e_i and h_i in (2.1) involve p_i. So it is not difficult to deduce that for an odd integer i, the coefficient of p_i in the expansion of P_i, when expressed as a polynomial in (odd) power sum symmetric functions, is equal to $1/i$.

The primitive subspace of $\mathbb{Z}[P_1, P_3, \ldots, P_{2n-1}]$ is spanned by $p_1, p_3, \ldots, p_{2n-1}$. It follows from Proposition 1 and the same argument as in Section 2.1 (but without the complication of the conjugation) that the map

$$\psi : \mathbb{Z}[P_1, P_3, \ldots, P_{2n-1}] \simeq H_*(\Omega\mathrm{Sp}(n)) \hookrightarrow H_*(\Omega\mathrm{Sp}(n+1)) \simeq \mathbb{Z}[P_1, P_3, \ldots, P_{2n+1}]$$

is the natural inclusion. From Theorem 1, we obtain the following corollary.

Corollary 2. *The symmetric function $P_w^{(n)}$ expands positively in terms of $\{P_v^{(n+1)}\}$. In particular, $P_w^{(n)}$ is positive in terms of Schur P-functions.*

2.3. $\mathrm{Sp}(2n,\mathbb{C})$ *to* $\mathrm{SL}(2n,\mathbb{C})$ *branching*

Let

$$\kappa : \mathbb{Z}[P_1, P_3, \ldots, P_{2n-1}] \simeq H_*(\Omega\mathrm{Sp}(n)) \hookrightarrow H_*(\Omega\mathrm{SU}(2n)) \simeq \mathbb{Z}[h_1, h_2, \ldots, h_{2n-1}]$$

be the Hopf-inclusion induced by Proposition 1. Since all our inclusions commute, to show that κ is the natural inclusion of symmetric functions, it suffices to show that $\kappa_\infty : \mathbb{Z}[P_1, P_3, \ldots] \simeq H_*(\Omega\mathrm{Sp}(\infty)) \hookrightarrow H_*(\Omega\mathrm{SU}(\infty)) \simeq \mathrm{Sym}$ is the natural inclusion. Let $\Gamma^* \subset \mathrm{Sym}$ denote the ring of symmetric functions dual to $\mathbb{Z}[P_1, P_3, \ldots]$ considered in [**10**]. We have $\Gamma^* = \mathbb{Z}[Q_1, Q_3, \ldots]$, where $Q_i = 2P_i$. The relations satisfied by Q_i can be deduced from [**10**, (2.8)].

The dual map $\theta : H^*(\Omega\mathrm{SU}(\infty)) \twoheadrightarrow H^*(\Omega\mathrm{Sp}(\infty))$ is given explicitly in [**13**, 3.18], where $H^*(\Omega\mathrm{SU}(\infty))$ is presented as $\mathbb{Z}[h_1, h_2, \ldots]$ and $H^*(\Omega\mathrm{Sp}(\infty))$ is presented as $\mathbb{Z}[Q_1, Q_3, \ldots]$, and the map is the surjection given by $\theta(h_i) = Q_i$ (the Schur Q-function Q_i is defined for even i as well). In terms of power sum symmetric functions, this map is given by $\theta(p_{2i}) = 0$ and $\theta(p_{2i+1}) = 2p_{2i+1}$. Thus, the dual κ_∞^* of our desired map differs from the map θ by Hopf-automorphisms: so we must have $\kappa_\infty^*(p_j) = \pm\theta(p_j)$. Taking duals and using [**10**, Lemma 2.1], which roughly says that $\theta : \mathrm{Sym} \to \Gamma^*$ and the inclusion $\mathbb{Z}[P_1, P_3, \ldots, P_{2n-1}] \subset \mathrm{Sym}$ are adjoint, we see that we have $\kappa_\infty(p_j) = \pm p_j$. To see that the sign is positive, we may argue as in Section 2.1, and deduce that κ_∞ itself is the natural inclusion of rings of symmetric functions. From Theorem 1, we obtain the following.

Corollary 3. *The symmetric function $P_w^{(n)}$ expands positively in terms of $(2n-1)$-Schur functions.*

With $n \to \infty$, we obtain as a special case the well-known fact that Schur P-functions are Schur positive.

3. *Combinatorial versus topological Hopf algebras*

Let H be a graded, connected, Hopf algebra defined over $\mathbb{Z}$. A character $\chi : H \to \mathbb{Z}$ is a morphism of $\mathbb{Z}$-algebras. Aguiar, Bergeron, and Sottile [**1**] define a *combinatorial Hopf algebra* to be a pair (H, χ). (The Hopf algebras of [**1**] are in fact over a field and we have changed to the integers.) The category $\mathcal{C}$ of combinatorial Hopf algebras has arrows $g : (H, \chi) \to (H', \chi')$ given by Hopf-morphisms $g : H \to H'$ such that $\chi = \chi' \circ g$. Symmetric functions Sym have a canonical character, given by $\chi_{\mathrm{Sym}}(h_i) = 1$, or $f(x_1, x_2, x_3, \ldots) \mapsto f(1, 0, 0, \ldots)$.

Aguiar, Bergeron, and Sottile show the following theorem.

THEOREM 2. *The terminal object of the category of cocommutative Hopf algebras is* $(\mathrm{Sym}, \chi_{\mathrm{Sym}})$.

From this point of view, the central thesis of this article, which is to express affine Schubert classes as symmetric functions, is very natural. The Hopf algebras considered in [**1**] have a combinatorial origin, while the Hopf algebras considered in the present article have a topological origin. It is thus natural to find the topological version of Theorem 2.

By a H-space we will mean a connected topological space X, which is equipped with a homotopy-associative multiplication $m : X \times X \to X$. A map of H-spaces is one that commutes with multiplication up to homotopy. Let us consider the category $\mathcal{T}$ the objects of which are pairs (X, E) where X is a H-space and E is a $U(\infty)$-bundle over X such that the following diagram is Cartesian:

$$\begin{tikzcd} E \times E \arrow[r] \arrow[d] & E \arrow[d] \\ X \times X \arrow[r] & X. \end{tikzcd} \tag{3.1}$$

The maps in $\mathcal{T}$ are given by homotopy classes of H-space maps which induce Cartesian diagrams. Let us denote the total Chern class of $U(\infty)$-bundle by $c(E)$. The element $c(E)$ lies in the completion of the cohomology $H^*(X)$, and gives rise to a linear map $\chi_E : H_*(X) \to \mathbb{Z}$. The condition that χ_E is a character is that $c(E)$ is grouplike: $m^*(c(E)) = c(E) \otimes c(E)$. This condition follows from the diagram (3.1).

The classifying space $BU(\infty)$ has a natural $U(\infty)$-bundle $EU(\infty) \to BU(\infty)$. The H-space structure $m : BU(\infty) \times BU(\infty) \to BU(\infty)$ classifies the Whitney sum of vector bundles. Thus if $E \to X$ and $E' \to X$ are classified by $f : X \to BU(\infty)$ and $f' : X \to BU(\infty)$ then $E \oplus E'$ is classified by $m \circ (f, f') : X \to BU(\infty) \times BU(\infty) \to BU(\infty)$. It follows that $(BU(\infty), EU(\infty))$ satisfies (3.1). We shall pick an isomorphism $H_*(BU(\infty)) \simeq \mathrm{Sym}$ so that $c(EU(\infty)) = 1 + h_1 + h_2 + \cdots$. (The usual isomorphism would give the elementary symmetric functions, so we compose with ω.) The following result follows nearly immediately from the definitions.

THEOREM 3. *The pair* $(BU(\infty), EU(\infty))$ *is the terminal object in the category* $\mathcal{T}$. *There is a functor* $\mathcal{T} \to \mathcal{C}$ *given by* $(X, E) \mapsto (H_*(X), \chi_E)$, *sending* $(BU(\infty), EU(\infty))$ *to* $(\mathrm{Sym}, \chi_{\mathrm{Sym}})$.

Proof. Let $(X, E) \in \mathcal{T}$. Let $f : X \to BU(\infty)$ be the (unique up to homotopy) map classifying the bundle $E \to X$. Then $f \times f : X \times X \to BU(\infty) \times BU(\infty)$ classifies $E \times E$ and the diagram

$$\begin{tikzcd} X \times X \arrow[r] \arrow[d] & BU(\infty) \times BU(\infty) \arrow[d] \\ X \arrow[r] & BU(\infty) \end{tikzcd} \tag{3.2}$$

is homotopy commutative by (3.1) for (X, E) and for $(BU(\infty), EU(\infty))$. Thus, the map $X \to BU(\infty)$ is a H-space map and $f : (X, E) \to (BU(\infty), EU(\infty))$ a morphism in $\mathcal{T}$. The last statement has already been established. $\square$

Remark 2. The terminal object in the category $\mathcal{C}$ of all combinatorial Hopf algebras is the Hopf algebra of quasisymmetric functions QSym. It is shown in [2] that QSym $\simeq H^*(\Omega\Sigma\mathbb{C}P^\infty)$, where Σ denotes suspension. Thus, there should be a different version of Theorem 3 with $\Omega\Sigma\mathbb{C}P^\infty$ as the terminal object.

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